MATH4060 Tutorial 2

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Problem 1. *Show that*

(a) *the order of growth of $f(z) = e^{z^2}$ is 2,*

(b) *the order of growth of $g(z) = \cos z^{1/2} = \sum_{n=0}^{\infty}(-1)^n z^n/(2n)!$ is 1/2,*

(c) $|\sin \pi z| \leq e^{\pi|z|}$.

(a) Using the power series expansion of the exponential function, we have $|e^{z^2}| \leq e^{|z|^2}$, so $\rho_f \leq 2$. On the other hand, for $x \in \mathbb{R}_{>0}$, if $e^{x^2} \leq Ae^{Bx^\rho}$, then $e^{x^2 - Bx^\rho}$ is bounded. Taking $x \to \infty$ shows that this is impossible when $\rho < 2$. Hence $\rho_f = 2$.

(b) We have

$$|g(z)| \leq \sum_{n=0}^{\infty} \frac{(|z|^{1/2})^{2n}}{(2n)!} \leq \sum_{m=0}^{\infty} \frac{(|z|^{1/2})^m}{m!} = e^{|z|^{1/2}},$$

so $\rho_g \leq 1/2$. On the other hand, for $-x \in \mathbb{R}_{<0}$,

$$g(-x) = \sum_{n=0}^{\infty} \frac{(x^{1/2})^{2n}}{(2n)!} = \cosh(x^{1/2}) = \frac{e^{x^{1/2}} + e^{-x^{1/2}}}{2} \geq \frac{1}{2} e^{x^{1/2}}.$$

An argument as in (a) shows that $\rho_g \geq 1/2$, hence, $\rho_g = 1/2$.

(c) We have

$$|\sin \pi z| = \frac{|e^{i\pi z} - e^{-i\pi z}|}{2} \leq \frac{e^{|i\pi z|} + e^{|-i\pi z|}}{2} = e^{\pi|z|}.$$

Problem 2 (Exercise 9)**.** *Let S be a sector at the origin forming an angle of π/β and F be a holomorphic function in S that is continuous on $\overline{S}$. Suppose that $|F(z)| \leq 1$ on ∂S and $|F(z)| \leq Ce^{c|z|^\alpha}$ for all $z \in S$, where $0 < \alpha < \beta$ and $C, c > 0$. Prove that $|F(z)| \leq 1$ for all $z \in S$.*

This is the extension of the Phragmén–Lindelöf principle to more general settings.

Without loss of generality, we may assume $S = \{re^{i\theta} : -\pi/(2\beta) < \theta < \pi/(2\beta)\}$ (otherwise just compose with a rotation). Consider $\alpha < \gamma < \beta$, and define, for $\epsilon > 0$,

$$F_\epsilon(z) = F(z)e^{-\epsilon z^\gamma}.$$

The condition on γ implies the following:

(i) $\gamma < \beta$: for $z = re^{i\theta} \in S$, we have $-\frac{\pi}{2} < -(\frac{\gamma}{\beta})\frac{\pi}{2} < \gamma\theta < (\frac{\gamma}{\beta})\frac{\pi}{2} < \frac{\pi}{2}$, and $\cos(\gamma\theta) \geq \delta > 0$, δ independent of θ.
So z^γ is well-defined using the principal branch of logarithm and $\operatorname{Re} z^\gamma = \operatorname{Re}(r^\gamma e^{i\gamma\theta}) = r^\gamma \cos(\gamma\theta) \geq r^\gamma \delta$. Thus

$$|e^{-\epsilon z^\gamma}| = e^{-\epsilon \operatorname{Re} z^\gamma} \leq e^{-\epsilon\delta r^\gamma}$$

(ii) $\alpha < \gamma$: for $z = re^{i\theta} \in S$, we have

$$|F_\epsilon(z)| \leq Ce^{cr^\alpha - \epsilon\delta r^\gamma} \to 0$$

as $|z| = r \to \infty$.

Because F_ϵ decays to zero near infinity, it must acquire a maximum at some $w \in \overline{S}$. By the maximum-modulus theorem, w cannot be in the interior unless $F \equiv 0$; now if $w \in \partial S$, the assumption of F on ∂S implies $|F_\epsilon(z)| \leq 1$ for all $z \in S$. Take $\epsilon \to 0^+$.

Problem 3 (Exercise 12, Hardy's Theorem)**.** *If f is a (continuous) function on $\mathbb{R}$ satisfying $f(x) = O(e^{-\pi x^2})$ and $\hat{f}(\xi) = O(e^{-\pi\xi^2})$, then f is a constant multiple of $e^{-\pi x^2}$.*

This theorem roughly says that the fastest f and $\hat{f}$ can simultaneously decay (unless being trivial) is when they are constant multiples of the Gaussian.

Step 1. $\hat{f}$ extends to an entire function: define

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x)e^{-2\pi izx}\,dx.$$

The proof that $\hat{f}(z)$ is holomorphic in S_a (for any $a > 0$) is similar to Theorem 3.1 (and other homework exercises). The key point is that the decay of $f(x)$ gives a quadratic term $-\pi x^2$ in the exponent that dominates the possible exponential growth from $e^{\mathrm{Re}(-2\pi izx)}$ in every given strip S_a.

Step 2. Assume $f(x)$ is even. Then $\hat{f}(z)$ is even. Define $g(z) = \hat{f}(z^{1/2})$. Show that g is entire and satisfies

$$|g(x)| \leq ce^{-\pi x} \quad \text{and} \quad |g(z)| \leq ce^{\pi|z|}$$

for $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

It is easy to see that $\hat{f}(-z) = \hat{f}(z)$ by changing the variable $x \mapsto -x$ in the definition and using that $f(x)$ is even. Note that $g(z)$ is originally well-defined and holomorphic on $-\pi < \theta < \pi$, but because $\hat{f}$ is even, $\hat{f}(iy) = \hat{f}(-iy)$, so $g(z)$ extends continuously along the branch cut. It is in fact holomorphic by e.g. the symmetry principle (Theorem 5.5, Chapter 2).

For $x \geq 0$, the first estimate follows from the original assumption $\hat{f}(\xi) = O(e^{-\pi\xi^2})$. For $x < 0$, this is the same as the second estimate, so we only need to prove the latter. In fact, for $z = re^{i\theta}$,

$$\begin{aligned}
|g(z)| &\leq \int_{-\infty}^{\infty} |f(x)|e^{\mathrm{Re}(-2\pi iz^{1/2}x)}\,dx \leq C\int_{-\infty}^{\infty} e^{-\pi(x^2-2x\,\mathrm{Im}\,z^{1/2})}\,dx \\
&= C\int_{-\infty}^{\infty} e^{-\pi(x^2-2xr^{1/2}\sin(\theta/2))}\,dx = Ce^{\pi r\sin^2(\theta/2)}\int_{-\infty}^{\infty} e^{-\pi(x-r^{1/2}\sin(\theta/2))^2}\,dx \\
&= C'e^{\pi r\sin^2(\theta/2)} \leq C'e^{\pi|z|},
\end{aligned}$$

where $C' > 0$ is independent of z.

Step 3. Take f and g as in Step 2. For $\beta > 1$, apply Phragmén–Lindelöf principle to the function

$$F_\beta(z) = g(z)e^{\gamma z} \quad \text{where } \gamma = \pi\frac{ie^{-i\pi/(2\beta)}}{\sin\pi/(2\beta)}$$

and the sector $0 \leq \theta \leq \pi/\beta < \pi$, and let $\beta \to 1^+$.

We need to bound F_β along the boundary and verify the growth estimate (with $\alpha = 1$). First, we have

$$|F_\beta(z)| \leq |g(z)|e^{|\gamma z|} \leq ce^{\pi|z|}e^{\pi|z|/(\sin\pi/(2\beta))} \leq ce^{3\pi|z|}$$

for β sufficiently close to 1. Along the positive real axis, the first estimate of Step 2 shows that

$$|F_\beta(x)| \leq ce^{-\pi x}e^{x\,\mathrm{Re}\,\gamma} = ce^{-\pi x}e^{x\pi} = c;$$

while for $z = Re^{i\pi/\beta}$, we have $\gamma z = iR\pi e^{i\pi/(2\beta)}/(\sin\pi/(2\beta))$, so $\mathrm{Re}(\gamma z) = -R\pi$ and

$$|F_\beta(z)| \leq ce^{\pi|z|}e^{-R\pi} = c.$$

Then the Phragmén–Lindelöf principle implies that $|F_\beta(z)| \leq c$. But $F_\beta(z) \to g(z)e^{\pi z}$ as $\beta \to 1^+$, and thus $g(z)e^{\pi z}$ is bounded on the closed upper half plane (including the negative real axis by continuity).

Repeat the same argument to the lower half plane with $-\pi < -\pi/\beta \leq \theta \leq 0$ and $\gamma = -\pi i e^{i\pi/(2\beta)}/(\sin\pi/(2\beta))$ to show that $e^{\pi z}g(z)$ is bounded in $\mathbb{C}$, hence a constant. Therefore $g(z)$ is a constant multiple of $e^{-\pi z}$ and $f, \hat{f}$ are multiples of $e^{-\pi z^2}$.

Step 4. For an odd function f, we similarly have $\hat{f}$ is odd, and so $\hat{f}(0) = 0$. Because $\hat{f}$ is entire and has a zero at 0, $\hat{f}(z)/z$ is entire and even. Repeating the above argument to this function (the estimates will continue to hold because $1/z$ will only add further decay near infinity) will show that $\hat{f}(z) = cze^{-\pi z^2}$, but this contradicts the assumption $\hat{f}(\xi) = O(e^{-\pi\xi^2})$ unless $c = 0$. So $f = \hat{f} = 0$.

Step 5. General case: write

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$